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COMMENT

An exact result for the magnetisation of the Kagomé lattice Ising model with magnetic field

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Abstract. We consider the anisotropic Kagomé lattice Ising model with three interaction parameters K_1, K_2, K_3 and a magnetic field H. It is shown that the magnetisation can be determined exactly for an appropriate relation between K_i and H.

In a recent letter, Giacomini (1988a)† derived exactly the partition function of the anisotropic Kagomé lattice Ising model for an appropriate relation between the interaction parameters K_i and the magnetic field H. We shall demonstrate that the magnetisation can also be derived exactly for the same relation. We shall follow his notation.

The magnetisation at the site denoted by σ_5 is

$$\langle\sigma_5\rangle_K = \left(\sum_{(\sigma)} \sigma_5 \prod W\right)[Z_{\text{Kag}}(K_1, K_2, K_3, H)]^{-1}. \tag{1}$$

The Kagomé lattice can be transformed into a decorated honeycomb lattice with interaction parameters M_1, M_2, M_3 such that

$$Z_{\text{Kag}}(K_1, K_2, K_3, H) = R^{2N} Z_{\text{dec}}(M_1, M_2, M_3, H) \tag{2}$$

$$\langle\sigma_5\rangle_K = \left(\sum_{(\sigma,S)} \sigma_5 \prod W'\right) Z_{\text{dec}}^{-1} \tag{3}$$

where

$$\begin{aligned} W'(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, S_1, S_2) &= \exp[S_1(M_1\sigma_3 + M_2\sigma_4 + M_3\sigma_5) + S_2(M_1\sigma_1 + M_2\sigma_2 + M_3\sigma_5) \\ &\quad + H\sigma_5 + \tfrac{1}{2}H(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)]. \end{aligned} \tag{4}$$

After the decimation of the spins σ_i, the decorated honeycomb lattice can be transformed into a honeycomb lattice with interaction parameters L_1, L_2, L_3 and a magnetic field $\bar{H}$ such that

$$Z_{\text{dec}}(M_1, M_2, M_3, H) = (A_1 A_2 A_3)^N Z_{\text{honey}}(L_1, L_2, L_3, \bar{H}). \tag{5}$$

It follows from the identity

$$\sum_{\sigma_5} \sigma_5 W' = \tanh[M_3(S_1 + S_2) + H] \sum_{\sigma_5} W' \tag{6}$$

that we have

$$\langle\sigma_5\rangle_K = \langle\tanh[M_3(S_1 + S_2) + H]\rangle_{\text{honey}} = a + b(\langle S_1\rangle + \langle S_2\rangle) + c\langle S_1 S_2\rangle \tag{7}$$

† A factor of $\sinh(2K_i)$ is missing on the second term in the RHS of equation (6) in this letter. The RHS of (15) should be $\alpha^2[\ldots]/4$.

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where

$$
\begin{aligned}
4a &= \tanh(2M_3+H)-\tanh(2M_3-H)+2\tanh H \\
4b &= \tanh(2M_3+H)+\tanh(2M_3-H) \qquad (8) \\
4c &= \tanh(2M_3+H)-\tanh(2M_3-H)-2\tanh H.
\end{aligned}
$$

At $H=0$, the site magnetisation (Naya 1951) and the nearest-neighbour spin–spin correlation (Baxter 1982) are known. At $H=\mathrm{i}\pi/2$, the magnetisation and the spin–spin correlation can be obtained from the corresponding expressions in zero field by the replacement (Lin and Wu 1988) $M_3 \rightarrow M_3+\mathrm{i}\pi/2$.

It is pointed out by Giacomini (1988b) that the partition function of the Ising model with a magnetic field $\bar{H}$ on a lattice with an even number of sites (e.g. the honeycomb lattice) remains invariant when $\bar{H}$ is transformed into $\bar{H}+\mathrm{i}\pi$. He proved that

$$Z_{\mathrm{Kag}}(K_1, K_2, K_3, H) = A^N Z_{\mathrm{Kag}}(K_1', K_2', K_3', H') \qquad (9)$$

where the Kagomé lattice with parameters K_i' and magnetic field H' can be transformed into a honeycomb lattice with a magnetic field $\bar{H}+\mathrm{i}\pi$. K_i' and H' are functions of K_i and H, and explicit expressions are given by Giacomini (1988b) for the isotropic case $K_1=K_2=K_3=K$. Unfortunately we cannot write down a similar relation for the magnetisation. We have

$$\langle\sigma_5\rangle_K' = a' + b'(\langle S_1\rangle + \langle S_2\rangle) + c'\langle S_1 S_2\rangle. \qquad (10)$$

Since $b/c \neq b'/c'$, we cannot eliminate both $(\langle S_1\rangle+\langle S_2\rangle)$ and $\langle S_1S_2\rangle$ by taking a linear combination of (7) and (10). When $H'=0$, we have $a'=c'=\bar{H}=0$ and it is well known that the magnetisation

$$\langle\sigma_5\rangle_K' = \tanh(2M_3')(\langle S_1\rangle + \langle S_2\rangle)/2 \qquad (11)$$

can be obtained from the known expression for the spontaneous magnetisation of the honeycomb lattice Ising model (Syozi 1972). Substituting (11) into (7), we get

$$\langle\sigma_5\rangle_K = a + 2b[\langle\sigma_5\rangle_K'/\tanh(2M_3')] + c\langle S_1 S_2\rangle_{\mathrm{honey}} \qquad (12)$$

where $\langle S_1S_2\rangle$ is the known nearest-neighbour spin–spin correlation of the honeycomb lattice Ising model at $\bar{H}=0$ (Baxter 1982). Consequently the magnetisation of the 3–12 lattice Ising model can be determined exactly when the condition $H'=0$, which is equivalent to equation (15) of Giacomini (1988a), is satisfied.

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